Code-aided SNR estimation based on expectation maximisation algorithm

N. Wu, H. Wang and J.-M. Kuang

A code-aided signal-to-noise ratio (SNR) estimator based on the expectation maximisation algorithm is proposed. The method iteratively uses the soft information from the channel decoder and significantly improves estimation precision in the low SNR regime. It can be extended to higher-order modulation such as MPSK and MQAM directly.

Introduction: Signal-to-noise ratio (SNR) estimation is an important problem in power control, turbo decoding, link adaptation, etc. Many papers have been published on SNR estimation over the AWGN channel [1–4]. In [1], a variety of data-aided (DA) and non-data-aided (NDA) SNR estimators are compared. In [2], a hybrid DA/NDA estimator jointly using pilots and data symbols is proposed to estimate transmission accuracy. An NDA SNR estimator based on the expectation maximisation (EM) algorithm is proposed in [3]. Symbol soft information feedback from the output of the demodulator is used to improve performance when SNR is low. All the estimators in [1–3] are designed for uncoded systems. In [4], an estimator, which is suitable to be embedded in a coded system with an iterative decoding scheme, is proposed to use a priori information of data symbols to improve estimation precision in the low SNR regime. However, this estimator is hard to be extended to higher-order modulation owing to the complexity of likelihood function.

In this Letter, we propose an SNR estimator derived from the EM algorithm for the coded system. Soft information of the data symbol from the channel decoder is used iteratively to maximise the likelihood function. Simulation results show a significant performance improvement in the low SNR regime over the other estimators which ignore the channel coding. In addition, the extension of the proposed estimator to higher-order modulation such as MPSK and MQAM is straightforward.

System model and ML SNR estimation: Consider the following discrete time complex-valued modulated symbol transmitted over an AWGN channel with perfect synchronisation

\[ y_k = A s_k + n_k \quad k = 1, 2, \ldots, K \]

where \( y_k \) is the sample of the matched filter output, \( A \) is a positive channel gain, \( s_k \in \mathcal{A} \) is the transmitted symbol from constellation set \( \mathcal{A} \), \( n_k \) is the noise sample of a complex AWGN with mean zero and variance \( \sigma^2 \).

Define a parameter vector \( \theta \in [A, \sigma^2] \). The SNR of the received signal can be defined as \( \gamma = A^2 / \sigma^2 \). The maximum likelihood (ML) estimation of parameter \( \theta \) is

\[ \hat{\theta}_{ML} = \arg \max_{\theta} \ln p(y | \theta) = \arg \max_{\theta} \ln \left( \sum_{s \in \mathcal{A}} p(y | s, \theta)p(s) \right) \]

where \( p(s) \) is the a priori probability of the symbol sequence transmitted. For the coded system, where \( A \) is a codeword, then

\[ p(s) = \begin{cases} 1/|\mathcal{A}|^R & \forall s \in \mathcal{B} \\ 0 & \forall s \in \mathcal{B}^c \end{cases} \]

where \( |\mathcal{A}| \) refers to the number of elements in set \( \mathcal{A} \), \( \mathcal{A}^4 \) is the set of modulated sequence, \( \mathcal{B} \subset \mathcal{A}^4 \) is the set of possible coded sequence, \( R \) is the code rate.

Code-aided SNR estimation based on EM algorithm: Substituting (3) in (2), we will have the code-aided (CA) ML SNR estimator. Unfortunately, a huge number of terms in the sum make the direct computation of (2) intractable. The EM algorithm is proposed to solve this problem iteratively.

Define \( y \) as the incomplete data set and \( z \equiv (y^T, s^T) \) as the complete data set. The EM algorithm proceeds in two steps: the expectation step (E-step) (4) and the maximisation step (M-step) (5):

\[ Q(\theta, \hat{\theta}^{(i-1)}) = E_z \left[ \ln p(z | y, \hat{\theta}^{(i-1)}) \right] = \sum_{s \in \mathcal{A}} p(s | y, \hat{\theta}^{(i-1)}) \ln p(z | \theta) \]

\[ \hat{\theta}^{(i)} = \arg \max_{\theta} Q(\theta, \hat{\theta}^{(i-1)}) \]

where \( p(y | \hat{\theta}^{(i-1)}) \) is the a posteriori probability of the transmitted sequence \( s \) conditioned on \( \hat{\theta}^{(i-1)} \), \( i \) refers to the \( i \)th iteration. \( \hat{\theta}^{(i)} \) converges to \( \theta_{ML} \) as long as the initial estimate is ‘close enough’. The log likelihood function (LLF) of the complete data \( z \) is

\[ \ln p(z | \theta) = \ln p(y | s, \theta) + \ln p(s) \]

Substituting (6) in (4) we have

\[ Q(\theta, \hat{\theta}^{(i-1)}) = \sum_{s \in \mathcal{A}} p(s | y, \hat{\theta}^{(i-1)}) \ln p(y | s, \theta) + \sum_{s \in \mathcal{A}} p(s | y, \hat{\theta}^{(i-1)}) \ln p(s) \]

Substituting (8) in (7) and dropping the terms independent of \( \theta \), we get

\[ \lambda(\theta, \hat{\theta}^{(i-1)}) = -K \ln \sigma^2 - \sum_{k=1}^{K} \ln |y_k|^2 + A^2 p_k - 2 \Re \{ \Im \{ \hat{\eta}_k \hat{\bar{\eta}}_k \} \} \]

where \( \eta_k \) and \( p_k \) are defined, respectively, as the a posteriori mean (10) and a posteriori mean square value (11) of the \( k \)th channel symbol \( s_k \) conditioned on \( \hat{\theta}^{(i-1)} \)

\[ \eta_k(y, \hat{\theta}^{(i-1)}) = \sum_{s \in \mathcal{A}} s_k p(y | s, \hat{\theta}^{(i-1)}) = \sum_{s \in \mathcal{A}} s_k p(s = s_k | y, \hat{\theta}^{(i-1)}) \]

\[ p_k(y, \hat{\theta}^{(i-1)}) = \sum_{s \in \mathcal{A}} |s_k|^2 p(s = s_k | y, \hat{\theta}^{(i-1)}) = \sum_{s \in \mathcal{A}} |s_k|^2 p(s = s_k | y, \hat{\theta}^{(i-1)}) \]

The partial derivatives of \( \lambda(\theta, \hat{\theta}^{(i-1)}) \) with respect to \( \theta \) are set to zero. After some algebra, we have the estimator expressed as

\[ \hat{\gamma}^{(i)} = \frac{K \left( \sum_{k=1}^{K} \Re \{ \Im \{ \hat{\eta}_k \hat{\bar{\eta}}_k \} \} \right)^2}{\left( \sum_{k=1}^{K} p_k \right)^2 \left( \sum_{k=1}^{K} \Im \{ \hat{\eta}_k \hat{\bar{\eta}}_k \} \right)^2 \sum_{k=1}^{K} \Re \{ \Im \{ \hat{\eta}_k \hat{\bar{\eta}}_k \} \}^2} \]

Multiplying a factor \((K–3)/K\) to (12) for a real-valued channel and a factor \((K–3)/2K\) for a complex-valued channel, we obtain the bias-reduced estimator [1]. This proposed novel estimator can be labelled as the EM-based iterative soft decision directed (EM-ISDD) SNR estimator, since it iteratively exploits the soft decision provided by the channel decoder. By replacing \( \eta_k \) in (12) with the corresponding hard decision directed (EM-IHDD) SNR estimator.

For QPSK modulation, \( p_k(y, \hat{\theta}^{(i-1)}) = 1 \) and \( \eta_k \) can be expressed as

\[ \eta_k(y, \hat{\theta}^{(i-1)}) = (1/\sqrt{2}) \left[ \begin{array}{c} \tan \left( A/\sqrt{2} \right) \right] \]

where \( [a^T, d^T] \) are the two bits Gray mapped onto the \( k \)th QPSK symbol, \( L(a^T) \) and \( L(d^T) \) are the corresponding log a posteriori probability ratio (LAPPR) obtained from the channel decoder. For higher-order modulation, the extension is tedious but straightforward, and is not presented here.

Simulation results and discussion: A Monte Carlo simulation of \( 10^4 \) trials is used to estimate the normalised bias (NBias) and normalised mean square error (NMSE), defined as in [1], associated with four estimators, namely, RxDA, M2M4 [1], EM-ISDD and EM-IHDD, in different SNRs. Consider an LDPC-coded QPSK system with block length \( n = 64800 \) and code rate \( R = 1/2 \). The maximum iterations for the LDPC decoder \( I = 20 \). For each estimator, \( K = 1024 \) data symbols are used for SNR estimation. EM-ISDD and EM-IHDD are initialised by a TxDA estimator [1] with 90 pilot symbols.

Figs. 1 and 2 illustrate the NBias and NMSE of the four SNR estimators. The normalised Cramer-Rao bound (CRB) [5] is also marked for comparison. We observe a threshold effect with the RxDA estimator when SNR is lower than 9 dB. This is due to the increase of symbol decision errors. The M2M4 estimator does not show any obvious threshold effect, but it cannot attach the CRB even in the high SNR regime.
EM-ISDD and EM-IHDD estimators perform very well when SNR $> 1$ dB. Their NMSE coincide with the CRB even by a single EM iteration. With further decrease of SNR, a threshold effect can be observed in both EM-ISDD and EM-IHDD estimators. This is because that SNR has entered the so-called ‘waterfall region’ of the LDPC code.

![Fig. 1 Normalised bias of four different SNR estimators](image1)

**Fig. 1** Normalised bias of four different SNR estimators

![Fig. 2 Normalised MSE of four different SNR estimators](image2)

**Fig. 2** Normalised MSE of four different SNR estimators

**Conclusion:** We propose a code-aided SNR estimator for the AWGN channel. The LAPPR from the output of the channel decoder is exploited iteratively by the EM algorithm to maximise the likelihood function. Simulation results demonstrate the significant performance improvement over other previously proposed estimators in respect of estimation precision. Moreover, extension of the proposed estimator to higher-order modulation is straightforward.

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**References**


